THE TWENTY-SEVENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The twenty-seventh annual meeting of the Society was held at Columbia University on Tuesday and Wednesday, December 28–29, 1920, extending through two sessions on each day. The attendance included the following eighty-six members:

Professor J. W. Alexander, Professor R. C. Archibald, Professor R. A. Arms, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor W. J. Berry, Mr. William Betz, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor R. W. Burgess, Professor B. H. Camp, Professor W. B. Carver, Professor F. N. Cole, Dr. G. M. Conwell, Professor J. L. Coolidge, Dr. W. L. Crum, Professor C. H. Currier, Dr. Tobias Dantzig, Professor J. V. De Porte, Dr. Jesse Douglas, Professor C. A. Fischer, Professor W. B. Fite, Professor C. H. Forsyth, Professor Tomlinson Fort, Mr. Philip Franklin, Dr. T. C. Fry, Professor R. E. Gilman, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor Olive C. Hazlett, Professor E. R. Hedrick, Dr. A. A. Himwich, Professor L. A. Howland, Professor W. A. Hurwitz, Mr. S. A. Joffe, Professor Edward Kasner, Dr. K. W. Lamson, Mr. Harry Langman, Professor Gillie A. Larew, Professor Florence P. Lewis, Professor P. H. Linehan, Professor Joseph Lipka, Mr. L. L. Locke, Professor W. R. Longley, Professor C. R. MacInnes, Professor H. F. MacNeish, Professor R. M. Mathews, Professor H. H. Mitchell, Professor Frank Morley, Dr. H. M. Morse, Professor G. W. Mullins, Professor G. D. Olds, Professor F. W. Owens, Dr. Helen B. Owens, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Dr. E. L. Post, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Dr. G. M. Robison, Dr. J. E. Rowe, Professor F. H. Safford, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Clara E. Smith, Professor P. F. Smith, Professor W. M. Smith, Professor Elijah Swift, Dr. J. S. Taylor, Professor H. D. Thompson, Mr. H. S. Vandiver, Professor Oswald Veblen, Professor J. N. Vedder, Professor J. H. M. Wedderburn, Professor H. S. White, Professor E. E. Whitford, Dr. Norbert Wiener, Professor A. H. Wilson, Professor Ruth G. Wood, Dr. T. S. Yang, Professor J. W. Young, Dr. S. D. Zeldin.

President Frank Morley occupied the chair, relieved by Professors G. D. Birkhoff and R. G. D. Richardson, on Tuesday; Professors Richardson and H. S. White presided at the sessions on Wednesday. The Council announced the election of the following persons to membership in the Society: Professor L. M. Coffin, Coe College; Professor I. H. Fenn, Polytechnic Institute of Brooklyn; Dr. Ludwik Silberstein, Eastman Kodak Company; Dr. W. L. G. Williams, Cornell University. One hundred twenty-one applications for membership were received.

A report was received by the Council from Professor E. R. Hedrick, chairman of the committee on increase of membership and sales of publications, presenting one hundred ten applications for membership in the Society and sixty-six subscriptions to the TRANSACTIONS. The report was accepted, with the thanks of the Council.

Professor T. S. Fiske, as representative of the contributors to the Bôcher memorial fund, tendered the fund to the Society to be held in trust and the income to be employed for the advancement of mathematical science. The trust was accepted, and a committee appointed to consider the most appropriate use to which the income of the fund could be devoted.

A committee was also appointed to make the necessary arrangements for the meeting of the Society to be held at Wellesley College in the summer of 1921.

The total membership of the Society is now 770, including 87 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 517; the number of papers read was 215. The number of members attending at least one meeting during the year was 306. At the annual election 189 votes were cast.

The Treasurer's report shows a balance of $8,994.53, including the Life Membership Fund, which amounts to $7,518.87. Sales of the Society's publications during the year amounted to $2,067.74.

The Library now contains 5,862 volumes, excluding some 500 unbound dissertations.

The afternoon session on Tuesday was especially marked by the retiring presidential address of Professor Frank Morley, on *Pleasant questions and wonderful effects*. A dinner was held at the Faculty Club Tuesday evening, at which fifty members were present.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

President, Professor G. A. BLISS.
Vice-Presidents, Professor F. N. COLE,
Professor DUNHAM JACKSON.
Secretary, Professor R. G. D. RICHARDSON.
Treasurer, Professor W. B. FITE.
Librarian, Professor R. C. ARCHIBALD.

Committee of Publication,
Professor E. R. HEDRICK,
Professor W. A. HURWITZ,
Professor J. W. YOUNG.

Members of the Council to serve until December, 1923,
Dr. T. H. GRONWALL, Professor FLORENCE P. LEWIS,
Professor O. D. KELLOGG, Professor A. D. PITCHER.

At the close of the morning session on Wednesday, Professor H. S. White in a short address tendered the thanks of the Society to Professor Cole for his distinguished services during his long term of office as secretary of the Society and editor of the BULLETIN. Professor White said in part:

"With the close of the year 1920, the resignation of Professor Frank Nelson Cole ends his long period of service as secretary of the American Mathematical Society. Since 1895 he has held that office, and since 1897 has been editor of the Society's BULLETIN, a periodical issued ten times a year, and has published all the Society's programs and announcements. From these onerous duties, and others, he now withdraws.

"With deep regret we acquiesce. But we desire to express to our retiring secretary our gratitude for his labors. He has dignified and amplified his office. With few precedents to guide him, he has met with energy and tact its problems. The meetings of the Society he has prepared and managed with skill and eminent success. Its membership he has aided greatly to enlarge, maintaining cordial and helpful relations with all. As editor he has made the BULLETIN increasingly interesting and useful, securing an expanding line of active contributors. He has maintained its high standards, while extending its range to keep pace with the augmenting activities of the Society.

"Surely fortunate is he who is able to sustain for twenty-

five years labors so varied and so strenuous; who can through steady, cumulative efforts develop and execute so fully his plans and policies. We extend to him our congratulations, in grateful recognition of his achievements; and we felicitate him now on his return to the engrossing problems of pure science."

The list of papers read at the meeting, together with abstracts of the papers, is as follows:

(1) Professor C. E. Wilder: *Einstein's four-dimensional space is not contained in a five-dimensional linear space.*

In 1880 Voss gave necessary and sufficient conditions that any curved manifold be contained in any higher-dimensional manifold. Using these conditions, Professor Wilder shows that any four-dimensional manifold that is contained in a five-dimensional linear space cannot satisfy the defining equations for an Einstein space.

(2) Dr. J. L. Walsh: *On the convergence of the Sturm-Liouville series.*

Dr. Walsh considers the differential system

$$u''(x) + [\rho^2 - g(x)]u(x) = 0 \qquad (0 \leqq x \leqq 1,\ u(0) = u(1) = 0), \tag{1}$$

a special case of which is the differential system

$$u''(x) + \rho^2 u(x) = 0 \qquad (0 \leqq x \leqq 1,\ u(0) = u(1) = 0), \tag{2}$$

whose normal solutions are the functions $\{\sin n\pi x\}$. If $f(x)$ is an arbitrary function integrable and with an integrable square, there is formed the series composed of the term-by-term difference of the formal expansions of $f(x)$ in terms of the normal solutions of (1) and of (2); it is proved that this series converges uniformly and absolutely to zero on the entire interval considered. The method of proof is entirely elementary.

(3) Miss Anna M. Mullikin: *Certain theorems concerning connected point sets.*

Miss Mullikin establishes the following theorems.

I. If in a plane S, m_1, m_2, m_3, $\cdots$ is a countable collection of closed, mutually exclusive point sets, no one of which disconnects S, and if m, the sum of these point sets, is closed, then m does not disconnect S.

II. If in a plane S, P and Q are two closed point sets with no point in common and H is a closed, bounded, connected set of points containing at least one point in common with P and at least one point in common with Q, then H contains a connected subset H_1, such that (1) H_1 contains no point in common with P or with Q, and (2) each of the point sets P and Q contains at least one limit point of H_1.

(4) Dr. A. R. Schweitzer: *On homogeneous functions as generators of an abstract field.*

Dr. Schweitzer points out that the algebra of logic, the theory of abstract groups, and the theory of abstract fields, as special disciplines on iterative compositions, so far as is known, have been first emphasized by him. In further development of the abstract field as an iterative theory, he considers the possibility of generating an abstract field by rational functions of m variables ($m \geqq 2$), (1) homogeneous and integral of degree $\geqq 2$, (2) homogeneous and rational of degree zero. In particular, he finds that the following functions may serve as undefined relations for an abstract field: $x^2 - yz$, $(x + y)\cdot z$, $(x \pm y)\cdot y$, and $x/y - 1$, the latter function being also quasi-transitive. In connection with the latter functions, it is worthy of note that the non-homogeneous functions $(1 \pm x)\cdot y$ generate a field under suitable postulational assumptions.

(5) Dr. A. R. Schweitzer: *The concept of an iterative compositional algebra.*

Dr. Schweitzer defines an iterative compositional algebra as a set E of elements $a_1, a_2, \cdots, b_1, b_2, \cdots$, etc., subject to the following assumptions. Let $\lambda_i(a_j b_j)$ ($i = 1, 2, \cdots, m$; $j = 1, 2, \cdots, n$) be generating relations of a given (formal or material, proper or improper) pseudo-group PG with a set (which may be vacuous) of adjoined conditions R on the functions λ_i; then the author defines

$$\Lambda_i(AB) = [\lambda_i(a_1 b_1)\cdots\lambda_i(a_n b_n)]$$

where A and B are n-ads in the sense defined by him in the AMERICAN JOURNAL OF MATHEMATICS (1909). Let the n-ads $\{A\}$ be elements of compositions $r_s[A_1, A_2, \cdots, A_p]$, $p \geqq 2$, $s = 1, 2, 3, \cdots$, where the r_s satisfy a given set of equations in iterative compositions in the sense previously defined by

the author. In particular, closure under the compositions r may exist. The relation of the compositions λ_i to r_s will be the subject of special inquiry; in general, the λ_i and r_s satisfy explicitly stated equations in iterative compositions of order $\leqq i + s$. More generally, for the pseudo-group PG, one might substitute any set of necessary properties of any postulationally defined calculus of iterative compositions. See, e.g., Schroeder, ARCHIV (2), vol. 5.

(6) Professor Joseph Lipka: *Transformations of trajectories on a surface.*

In a paper read at the October meeting of the Society, Professor Lipka proved five geometric properties which completely characterize the trajectories on a surface for any positional field of force. In the present paper, a study is made of the point transformations which convert systems of curves on a surface S possessing some or all of these properties into like systems on the transformed surface S_1. The general results are that systems of curves possessing the first property or the first two properties are invariant under an arbitrary point transformation, while all other systems are only invariant under a geodesic transformation. A similar study is made for the point transformations of "n"-systems—brachistochrones, catenaries, velocity curves, etc.—on a surface.

(7) Mr. Harry Langman: *Conformal transformations of period n and groups generated by them.*

Professor Kasner has studied the groups of transformations generated by conformal transformations of period 2.

Mr. Langman's paper is principally concerned with a generalization of a problem studied by Kasner. The generalized problem, following Kasner's notation, consists in finding whether two transformations $f(z)$ and $g(z)$ can always be found such that $f_m(z) = g_n(z) = z$ and $g[f(z)] = F(z)$ identically for some integral values of m and n. The subscripts here denote the iterated functions of corresponding order. In the more general problem no such numerical relation as that found by Kasner for the case $m = n = 2$ is found necessary (other than $K_1{}^r = 1$). The complete result states that if $K_1 \neq 1$, then $F(z)$ is always factorable; if $K_1 = 1$, the periods of the factor transformations, if they exist, must be equal; if $K_1 = 1$ and $K_2 \neq 0$, then $F(z)$ is always factorable into two trans-

formations of arbitrary period > 2; if $K_1 = 1$, $K_2 = K_3 = \cdots = K_r = 0$, and $K_{r+1} \neq 0$, then $F(z)$ can not be factored into transformations of order r, or any factor of r, but can always be factored into transformations of any other order > 2.

In the first part of the paper it is shown that if the transformation $F(z)$ is of period n, then it may be put uniquely into implicit form. The well known fact that $F(z)$ is conformally equivalent to a rotation about the origin is then readily deduced. The method also gives complete solutions of similar functional equations of the symbolic forms: $f^m = g$, where g is given and $g^n = 1$; $1 + f + f^2 + \cdots + f^{n-1} = 0$; etc. It is also shown that, corresponding to every function f, where $f^n = 1$, there are others, F, satisfying the last functional equation.

Kasner has considered other transformations of period 2, termed by him *conformal symmetries*, defined by $f(z_0)$, where z_0 is the conjugate of z. It is here shown that no other kind of transformation of this type exists, i.e. that all *reverse* conformal transformations are of period 2.

(8) Professor O. E. Glenn: *On a new treatment of theorems of finiteness. Second paper.* (Preliminary report.)

Professor Glenn's paper consists, in its present form, of two parts. A third part, dealing with certain amplifications, is to be added. The first part comprises a proof of the finiteness of binary formal modular concomitant systems (modulo p), which has been an outstanding unsolved problem since the publication of Dickson's first paper on modular invariants (1907) and of a paper by A. Hurwitz (1903). This proof is based upon the algorism on modular concomitants appertaining to domains mentioned in the present author's memoir in the TRANSACTIONS for 1920. The second part of the paper gives statements and solution of certain finiteness propositions in the realm of differential invariants. The present paper is, in regard to general method, a sequel to the one published under the same title in the TRANSACTIONS for 1919.

(9) Professor J. E. Rowe: *The efficiency of projectile and gun.*

The purpose of this investigation is to formulate mathematically a means by which the efficiencies of different pro-

jectiles fired in different guns may be compared, using as a criterion the energy wasted in propelling the projectile to the point of fall. It is possible also to obtain from this the economic efficiency, by taking into account the cost of the material used.

(10) Dr. S. D. Zeldin: *On the structure of finite continuous groups with one two-parameter subgroup.*

This paper considers groups with one two-parameter subgroup. By imposing certain conditions on groups meroedrically isomorphic with the given ones, the author shows how the structure can be simplified.

(11) Dr. S. D. Zeldin: *On the structure of finite continuous groups with a finite number of exceptional infinitesimal transformations.*

In a previous paper, presented to the Society in December, 1919, Dr. Zeldin discussed the structure of finite continuous groups having one exceptional infinitesimal transformation. The present paper deals with the structure of groups having any finite number of exceptional transformations.

(12) Mr. H. S. Vandiver: *On quadratic congruences and the factorization of integers.*

In this paper, Mr. Vandiver considers the problem of the practical determination of the integral values x in the congruence $x^2 \equiv a \pmod{m}$ where a and m are rational integers prime to each other, a positive. A method of trial and exclusion is given, based on the theory of the quadratic form $u^2 - av^2$, which supplements another method due to H. J. S. Smith (*Collected Papers*, vol. 1, p. 148), who used a definite in lieu of an indefinite quadratic form. A similar procedure is convenient in finding the factors of an integer n, if n can be expressed in the form $y^2 - kz^2$, where k is a small positive integer.

(13) Professor E. V. Huntington: *A mathematical theory of proportional representation.*

When N representatives are distributed among the several states, the *true quotas* are usually fractional, and are necessarily replaced by whole numbers. Various methods of minimizing the injustice involved in these replacements have been proposed (by Willcox, Hill, d'Hondt, etc.), but without satis-

factory mathematical analysis. Professor Huntington shows that in the case of two states the *logarithmic error*, $\log(x/y)$, where x = the ratio of the assignments as they actually are, and y = the ratio of the assignments as they ought to be, provides a complete criterion for the *best* apportionment. If there are more than two states, the following principle is proposed: If the injustice belonging to any pair of states can be reduced by a transfer of representatives within that pair, this improvement should be made. It is shown that in any given case there is just one apportionment which is incapable of such improvement, and may therefore be regarded as the *best*. It is also shown how this *best* apportionment may be very easily found, by the use of a *priority list* formed by multiplying the population of each state by the series of factors

$$\infty,\ 1/\sqrt{1\cdot 2},\ 1/\sqrt{2\cdot 3},\ 1/\sqrt{3\cdot 4},\ \cdots,$$

and arranging the results for all the states in order of magnitude.

(14) Dr. H. M. Morse: *Recurrent motions of the discontinuous type.*

In investigating the nature of the trajectories in a stable dynamical system for unlimited values of the variable which represents the time, Professor Birkhoff has shown that one is led to types of motions which he calls recurrent motions, and which he divides into discontinuous and continuous types. Simple examples of motions of the continuous type can be given. The present paper by Dr. Morse offers the first proof of the existence of recurrent motions of the discontinuous type for the case where the motion is represented by a curve with a continuously turning tangent. This proof is the outcome of a complete classification, from the point of view of analysis situs, of geodesics on surfaces of negative curvature. The paper will be published in the TRANSACTIONS.

(15) Professor Frank Morley: Presidential Address: *Pleasant questions and wonderful effects.*

President Morley's address will appear in full in the April number of the BULLETIN.

(16) Professor Edward Kasner: *Properties of orbits in the general theory of relativity.*

The orbits in the Einstein theory are the geodesics of the

four-dimensional manifold M_4 defined by $ds^2 = \Sigma g_{ik} dx_i dx_k$. Professor Kasner considers an arbitrary map of M_4 upon a 4-flat (x, y, z, t). The properties of the ∞^6 curves mapping the geodesics are studied. These curves are then projected orthogonally upon a 3-flat (x, y, z). The ∞^1 orbits corresponding to a given world-point (x, y, z, t) and a given direction $(dx : dy : dz)$ are shown to have the property that the locus of the centers of curvature is the inverse of a conic. If in particular the ds^2 is of the *static* form, studied by Levi-Civita and others, so that the world splits up into ordinary time and space, then it is shown that the conic becomes a circle through the given point; this means that the Meusnier property (relating to the variations of the curvature and the osculating plane) holds. In the Newtonian dynamics of a positional field of force, the locus becomes merely a straight line, since the ∞^1 trajectories then have a common osculating plane. Some further properties obtained for the general case are too complicated for brief statement.

(17) Professor Edward Kasner: *The solar gravitational field in finite form.*

In the last of his five notes on the Einstein theory presented at the summer meeting, Professor Kasner showed that there are no four-dimensional manifolds obeying Einstein's equations $G_{ik} = 0$ which can be immersed in a 5-flat; but that for a 6-flat there exist an infinity of solutions, including in particular the solar field. He now examines this field, taken in the Schwarzschild form, in detail. The final result is

$$ds^2 = -dx^2 - dy^2 - dz^2 + dX^2 + dY^2 - dZ^2,$$

where

$$X = \sqrt{\frac{r - 2m}{r}} \sin t, \qquad Y = \sqrt{\frac{r - 2m}{r}} \cos t,$$

$$Z = \sqrt{m} \sqrt{\frac{2r^3 + m}{r^3(r - 2m)}}\, dr,$$

where m is the mass of the sun, and $r^2 = x^2 + y^2 + z^2$. We thus have in finite form a model of the solar field. The model is situated in a flat space of six dimensions; or, using the more exact terminology of Weyl, in an affine-euclidean space of $(2 + 4)$ dimensions.

(18) Dr. Norbert Wiener: *The average of an analytic functional.*

Dr. Wiener discusses in this paper the notion of the average value of a functional which can be expanded in a series of multiple integrals, and develops an average which, over a considerable range of cases, is identical with that treated in his other papers, but which will apply directly to functionals that are not bounded.

(19) Dr. Norbert Wiener: *The average of a functional.*

Dr. Wiener gives a definition of the average value of a functional which reduces to a special case of the type of Daniell integral already treated by him in his paper entitled *The mean of a function of arbitrary elements*, presented to this Society at its meeting in December, 1919. The average thus developed is in intimate relation with the theory of probabilities, for it is based on the assumption that the distribution of the values of $F(T_1 + T_2)$, where F ranges over all continuous functions such that $F(T_1)$ has a certain determinate value X_1, is what is known statistically as a normal distribution and is independent of T_1 and X_1.

(20) Dr. Norbert Wiener: *Further properties of the average of a functional.*

Dr. Wiener defines the average of the bounded continuous functional F as

$$A\{F\} = \lim \pi^{-n/2} \prod^{n} (x_k - x_{k-1})^{-1/2} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} F\{\{\Psi\}(x)\}$$

$$\times \exp\left\{ -\frac{y_1^2}{x_1^2} - \frac{(y_2 - y_1)^2}{x_2 - x_1} - \cdots - \frac{(y_n - y_{n-1})^2}{x_n - x_{n-1}} \right\} dy_1 \cdots dy_n,$$

where $x_0, \cdots, x_n$ is a set of numbers satisfying the conditions

$$x_0 = 0, \qquad x_n = 1, \qquad x_{k+1} > x_k \text{ for every } k,$$

and where

$$\{\Psi\}(x) \equiv \{\Psi(x_1, \cdots, x_n)(y_1, \cdots, y_n)\}(x)$$

is that function of x which for $x_k < x < x_{k+1}$ assumes the value

$$y_k + \frac{x - x_k}{x_{k+1} - x_k}(y_{k+1} - y_k) \qquad (y_0 = 0).$$

The operation *lim* refers to the limit taken as the x's fill the interval from 0 to 1 more and more closely.

Dr. Wiener shows that $A\{F\}$ exists for every bounded continuous F. He shows, moreover, that the notion of average defined here is capable of a Daniell extension which is identical with the notion of average defined in the preceding paper.

(21) Professor Gillie A. Larew: *The Hilbert integral and Mayer fields for the problem of Mayer in the calculus of variations.*

A. Mayer and Bolza have shown that it is possible to extend to the problem of Lagrange in the calculus of variations the Hilbert theory in which an invariant integral is utilized in the proof that a certain set of conditions is sufficient for the existence of an extremum. A similar extension of this theory to the problem of Mayer is the subject of Professor Larew's paper. An integral is constructed effective in establishing a Weierstrass theorem, and it appears that the imposing of conditions for this integral to be independent of the path is equivalent to placing on the field of extremals conditions analogous to those characterizing the *Mayer fields* well known to students of the Lagrange problem. The study is confined to a Mayer problem in non-parametric representation and with fixed end-points. It may be noted that Kneser, using other methods, has found results consistent with these for the same problem in parametric form.

(22) Professor R. M. Mathews: *Generalizations of the classical construction of the strophoid.*

In the classical construction of the strophoid the curve is defined as the locus of the intersection of a pencil of circles with a pencil of lines. The circles have their centers on a line l and all pass through a point O on l. Each circle corresponds to that line of the pencil (F) which passes through its center. O is the node, F is the singular focus, and l is parallel to the asymptote. Professor Mathews generalizes this construction in two ways: first, for a pencil of circles through the node with their centers on any line through that point; and second, for a pencil of circles through any two points whose tangents meet on the curve.

(23) Professor W. A. Hurwitz: *Some properties of methods of evaluation of divergent sequences.*

In case a linear transformation, $y_n = \Sigma_{k=1}^{\infty} a_{nk}x_k$, carries every convergent sequence (x_n) into a sequence (y_n) converging to the same value as (x_n), the transformation is said to be regular; necessary and sufficient conditions for regularity are furnished by the Silverman-Toeplitz theorem. In the present paper, Professor Hurwitz investigates the conditions under which a regular transformation of real elements maintains the property of regularity even in the case of the improper limits $\pm\infty$, and also the conditions under which it carries a divergent sequence into another which, if it is not convergent, at least has its limits of oscillation not further separated than in the original sequence. Application is made to the particular case of *analytically regular* transformations, as defined by Hurwitz and Silverman.

(24) Professor W. C. Graustein: *Parallel maps of surfaces.*

Two real surfaces, in continuous one-to-one point correspondence, with the directed normals parallel at corresponding points, are said to correspond by a parallel map; or more specifically, by a directly parallel or an inversely parallel map, according as corresponding directions of rotation about corresponding points are the same or opposite. In Professor Graustein's paper, parallel maps are further classified as hyperbolic, elliptic, or parabolic, after the manner of the classification of one-dimensional projective correspondences, and each parallel map is characterized by an invariant analogous to the invariant of such a correspondence. The classification of parallel maps thus obtained is instructive in the light it sheds on known theorems concerning parallel maps, and it proves to be of value in developing new results.

(25) Professor J. H. M. Wedderburn: *On the maximum value of a determinant.*

Professor Wedderburn gives a proof, due to J. Schur (MATHEMATISCHE ANNALEN, vol. 66 (1909), pp. 488–510), of Hadamard's theorem that the maximum value of the determinant of a matrix $A = (a_{rs})$ is $|A| \leqq \Sigma(|a_{rs}^2|/n)^{n/2}$. As this inequality was not the main purpose of Schur's paper, his exposition does not show how brief the proof really is. If $\bar{a}_{rs}$ denotes the conjugate imaginary of a_{rs} and $\bar{A} = (\bar{a}_{rs})$, then $\bar{A}'A$ is a positive Hermitian form whose roots, as is well known, are real and positive. The sum of these roots is $\Sigma_{rs}a_{rs}\bar{a}_{rs} = \Sigma|a_{rs}^2|$;

hence their product is less than or equal to $(\Sigma|a_{rs}^2|/n)^n$. This product is equal to the determinant of $A'\bar{A}$ which is equal to the absolute value of $|A|^2$, thus proving the required inequality.

(26) Professor J. H. M. Wedderburn: *On the automorphic transformation of a bilinear form.*

The problem of the automorphic transformation of a bilinear form is usually discussed on the basis of the Cayley-Hermite solution. This solution is rational but becomes indeterminate under certain conditions. Professor Wedderburn derives a very simple solution in transcendental form which includes the exceptional cases and reveals their analytical nature clearly.

(27) Professor J. W. Lasley, Jr.: *Some special cases of the flecnode transformation of ruled surfaces.*

The process of obtaining the flecnode surface of a ruled surface leads to the consideration of a suite of surfaces, called the flecnode suite. Questions arise as to the cases in which this suite terminates or returns into itself. It is the purpose of Professor Lasley's paper to study some of these questions. It is found that the suite terminates with its first transform if and only if the given ruled surface has a straight line directrix. In case the termination occurs with the second transform, both branches of the flecnode curve on the given ruled surface may be obtained without integration. The suite is of period two if and only if the flecnode curve meets every generator in two coincident points. The flecnode suite cannot be of period three, nor of period four. The conditions for termination and periodicity considered are expressed in terms of the invariants of the differential equations which define the given ruled surface.

(28) Professor R. G. D. Richardson: *The theory of relative maxima and minima of quadratic and hermitian forms and its application to a new foundation for the theory of bilinear forms. First paper: Equivalence of pairs of bilinear forms.*

The theory of maxima and minima, which plays an ever increasing rôle in analysis, suggests a simpler and direct method for the discussion of a pair of forms. In the case of quadratic or hermitian forms, one of which is definite, each zero of the λ-determinant can be interpreted as a maximum

of a certain isoperimetric problem and as a minimum of another. This point of view enables Professor Richardson in this simplest case to obtain a direct method for reducing the pair of forms to the normal type as follows. Denoting by $\lambda_1, \cdots, \lambda_n$ the zeros of the λ-determinant, and setting $\lambda = \lambda_1$ in the corresponding λ-matrix $||a_{ij} - \lambda b_{ij}||$, the rows are regarded as the coefficients of linear homogeneous equations and the solution used as the first row of a matrix. The other rows are obtained in the same way by setting in the matrix $\lambda = \lambda_2, \cdots, \lambda_n$. The elements of the columns also serve as coefficients of linear equations which determine a second matrix. In the case of quadratic forms, this is identical with the first. When the λ-matrix is multiplied by these two matrices it is reduced to one with terms in the main diagonal only.

By a generalization of this process suggested by the formal work in the problem of taking the relative extreme, any pair of non-singular bilinear forms is reduced to its normal type. When the multiplicity of any λ_i is greater than its index (the number of linearly independent solutions of the corresponding linear equations) some of the rows and columns of the multiplying matrices are solutions of non-homogeneous equations extended from the original homogeneous equations by a process resembling the taking of a derivative.

The equivalence of pairs of bilinear forms is then based on three simple notions: (1) the equivalence of polynomials in λ; (2) the equality of the indexes of λ_i; (3) the possibility of solution of certain non-homogeneous linear equations.

(29) Dr. J. S. Taylor: *The analytic geometry of complex variables with some applications to function theory.*

Dr. Taylor's paper lays the foundations for a development of the analytic geometry of complex variables and applies some of the results to a geometric interpretation of function theory. A pair of complex numbers X and Y is represented by a point in four-space. Several definitions of distance are discussed, a geometric meaning is given to a complex angle, and six new pseudo-trigonometric functions of a complex angle are defined. The surfaces representing equations of the first and second degree in two complex variables are investigated in detail. Finally, a comparatively simple geometric meaning is given to the derivative of a function of a single complex variable, to its differential, and to its integral.

(30) Professor C. H. Forsyth: *The value of a bond to be redeemed ultimately, both principal and interest, in equal installments.*

The formulas for computing the value of a bond to yield a specified rate of interest have been derived for all cases where the principal is to be redeemed in one sum or in equal installments. The formula has been derived for the case where both principal and dividend are to be redeemed in equal annual installments. Professor Forsyth derives the formula for the more general case where the principal and dividend are to be redeemed in equal installments, but where the first installment is to be repaid at the end of f years and the rest at intervals of t years.

(31) Professor C. H. Forsyth: *Valuation of bonds bought to realize a specified rate of interest, assuming the amortizations to accumulate at a savings-bank rate.*

Investors in bonds realize only too well the usual impossibility of obtaining the rate of interest corresponding to the price computed by the ordinary formulas simply because the amortizations are relatively small sums, and small sums can rarely demand as high rates of interest as can such large sums as the principals of the bonds. Professor Forsyth derives a formula for computing the price of the conventional bond to yield actually a specified rate of interest since the amortizations are assumed to accumulate at a practical rate of interest, such as that of a savings bank. The formula thus derived proves singularly convenient for the solution of the inverse problem, which is the determination of the rate of interest that a given price will yield. In fact, the solution of the inverse problem is almost as easy and quite as simple as that of the direct problem.

(32) Dr. Einar Hille: *Zeros of Legendre functions.*

In this paper Dr. Hille considers solutions of Legendre's differential equation

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + a(a + 1)w = 0$$

with respect to the location of their zeros in the complex plane. The distribution of zeros along rays from $z = +1$ is investigated by means of two fundamental integral equalities which

are generalizations of well known formulas generally attributed to Green. It is proved that $P_a(z)$, the solution regular at $z = +1$, has its zeros in the interval $(-1, +1)$ when a is real, and in the interval $(+1, +\infty)$ when $a = -\frac{1}{2} + \mu i$. The number of zeros has been previously determined by Dr. Hille. When $a = \lambda + \mu i$ $(\lambda > -\frac{1}{2}, \mu > 0)$ it is proved that the zeros lie in a certain sector of the upper half of the plane. Finally, the solution $Q_a(z)$ which belongs to the exponent $1 + a$ at infinity is shown to have no zeros in the finite plane as long as $\lambda > -\frac{1}{2}$.

(33) Professor W. B. Carver: *Systems of linear inequalities.*

In a paper on *Systems of linear inequalities* (ANNALS OF MATHEMATICS, vol. 20, p. 191, March, 1919), Professor L. L. Dines gives a necessary and sufficient condition for the existence of solutions of such a system. His conditions are expressed in terms of the *I*-rank of the matrix, the definition of *I*-rank being somewhat analogous to that of rank.

In the present paper Professor Carver gives a necessary and sufficient condition that solutions of the system should *not* exist, the condition being expressed in a form similar to that ordinarily used to define linear dependence. Independence of a system and equivalence of two systems are then defined, and necessary and sufficient conditions found for them.

(34) Professor J. L. Coolidge: *Differential geometry of the complex plane.*

Professor Coolidge's paper is devoted to the study of systems of points in the complex plane, whose coordinates are analytic functions of two real variables, yet which do not lie on a real curve. Some attention is also paid to three-parameter systems.

(35) Professor C. L. E. Moore: *Note on minimal varieties in hyperspace.*

Professor Moore's paper appeared in the February number of this BULLETIN.

(36) Professor I. J. Schwatt: *Independent expressions for the Bernoulli numbers.*

Laplace, Stern, Kronecker, Shovelton and others have developed independent expressions for the Bernoulli numbers. Professor Schwatt has established methods which have

enabled him to find new independent expressions for these numbers. The methods are believed to be more direct and the results more simple than those obtained heretofore.

(37) Professor I. J. Schwatt: *Relations involving the numbers of Bernoulli and Euler.*

By the methods of the preceding paper, Professor Schwatt has developed new relations involving the numbers of Bernoulli and Euler.

(38) Professor I. J. Schwatt: *Independent expressions for Euler numbers.*

Very few independent expressions for the Euler numbers have been developed. The simplest among them is the one by Worpitzky (JOURNAL FÜR MATHEMATIK, vol. 94, p. 203). The expressions by Scherk (ibid., vol. 4, p. 299), and by Saalschütz (*Vorlesungen über die Bernoullischen Zahlen*, p. 100) are rather complicated. By means of operations with series Professor Schwatt has been enabled to devise methods and to obtain results which are believed to be quite simple.

(39) Professor I. J. Schwatt: *Independent expressions for the Euler numbers of higher order.*

Lucas (BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 6, p. 53) has introduced the Euler number of higher order. Professor Schwatt has been able to find only two independent expressions representing these numbers, one by Ely (AMERICAN JOURNAL OF MATHEMATICS, vol. 5, p. 339) and the other by Shovelton (QUARTERLY JOURNAL OF MATHEMATICS, vol. 46, p. 220). By means of operations with series methods have been devised which lead to simple independent expressions of these numbers. Three independent expressions for T_{np}, the coefficient of $x^{2n+p}/(2n+p)!$ in the expansion of $\tan^p x$, are also given.

(40) Professor I. J. Schwatt: *Summation of a type of Fourier's series.*

The principles used in the direct summation of

$$\sum_{n=0}^{\infty}(-1)^n \frac{\sin(a+ng)}{b+nh} r^n,$$

when r may be 1, in which case $g \equiv 0 \pmod{2\pi}$, is first applied to $\sum_{n=0}^{\infty}(-1)^n[1/(b+nh)]$. The method given by Professor

Schwatt in this paper is more general in its application than the one which he has presented in an article in the NOUVELLES ANNALES DE MATHÉMATIQUES, (4), vol. 6, pp. 203–209 (May, 1916).

(41) Professor F. W. Owens: *On the projectivity assumption in projective geometry.*

Various forms of the projectivity assumption in projective geometry have been made by different writers. A number of these are given and shown to be equivalent in Veblen and Young's *Projective Geometry.* In this paper, Professor Owens derives the ordinary forms of statement from milder forms of the assumption.

(42) Professor R. W. Burgess: *On certain simple skew frequency curves.*

To facilitate the discussion of frequency distributions which do not conform to the normal probability curve, Pearson in 1895 obtained a series of skew frequency curves by integrating the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a + bx}{c + dx + ex^2}.$$

This equation is a generalization of the slope property of the normal curve

$$\frac{1}{y}\frac{dy}{dx} = -ax.$$

The aim of the present paper is to give, by considerations based on the elementary theory of probability, a new basis for some of Pearson's curves, and to propose modifications in his methods of fitting these curves to given data. Professor Burgess' point of view is that the individual items summarized by the frequency curve may be regarded as being deviations all in one direction from a fixed origin, and, in the discrete case, as being built up by successive increments, the probability of adding each increment being variable. Among the curves derived on this basis are those of the family $y = cx^n e^{-kx}$. These curves are one type of those introduced by Pearson, and are also cases of Knibbs' *flexible curve.*

In view of the fundamental importance of the area in work with frequency curves, Professor Burgess proposes to use, as the criterion for goodness of fit, the sum of the differences of

area from the origin to various values of x, between the actual frequency distribution and the trial curve. The trial curves for this method are obtained by the use of a table giving the percentage of area to the right of the centroid for important values of n. This method requires the construction of tables of areas of these curves, that is, of the integral

$$I(x, n) = \int_0^x x^{n-1}e^{-x}dx/\Gamma(n).$$

Such tables may be easily constructed for values of n which are integral multiples of one-half by the use of the difference equation

$$I(x, n+1) - I(x, n) = -y(x, n+1)$$

and of tables of the exponential and the error functions.

(43) Dr. G. M. Robison: *Divergent double series and sequences.*

For any given double sequence s_{mn}, a new double sequence σ_{mn} may be defined by a linear transformation:

$$\sigma_{mn} = \sum_{k=1,\, l=1}^{m,\, n} a_{mnkl} \cdot s_{kl}.$$

Dr. Robison establishes a necessary and sufficient condition that the transformation be regular, i.e., that when it is applied to a bounded convergent sequence the new sequence shall be convergent to the same value as the original sequence. Further investigations determine when a regular transformation carries a sequence which becomes positively infinite into a sequence which becomes positively infinite. The condition that a transformation transform a bounded convergent sequence (s_{mn}) into a bounded convergent sequence (σ_{mn}), the limits not necessarily being the same, is given; also the condition that a transformation carries a bounded sequence into a bounded convergent sequence. These theorems are also extended to the case where the new sequence is defined as follows:

$$\sigma_{mn} = \sum_{k=1,\, l=1}^{\infty,\, \infty} a_{mnkl} \cdot s_{kl},$$

each element σ_{mn} depending upon all the elements of the original sequence.

(44) Professor G. D. Birkhoff: *An extension of Poincaré's geometric theorem.*

The hypothesis of Poincaré's geometric theorem* requires that the two boundaries of a ring be invariant. As Professor Birkhoff shows, it is sufficient to require the invariance of only one of these boundaries. The extension is very convenient for the dynamical applications.

(45) Dr. J. L. Walsh: *On the location of the roots of polynomials.*

Dr. Walsh proves that if the points $a_1, a_2, \cdots, a_k$ lie on or within a circle whose center is α and radius r_1 and if the points $b_1, b_2, \cdots, b_k$ lie on or within a circle whose center is β and radius r_2, then all the roots of the polynomial

$$f(z) = (z - a_1)(z - a_2)\cdots(z - a_k) - A(z - b_1)(z - b_2)\cdots(z - b_k), \qquad A \neq 1,$$

lie on or within the k circles whose common radius is

$$(r_1 + |A^{1/k}|r_2)/(1 + |A^{1/k}|)$$

and whose centers are the k points $(\alpha - \beta A^{1/k})/(1 - A^{1/k})$, where $A^{1/k}$ takes all the k values possible. If any one of these k circles is exterior to all the others, that circle contains precisely one root of $f(z)$. This theorem holds with a very slight change if $A = 1$. There is a similar theorem for the polynomial $(z - a_1)(z - a_2)\cdots(z - a_k) - A = 0$.

Miss Mullikin was introduced by Professor R. L. Moore, and Dr. Hille by Professor G. D. Birkhoff. In the absence of the authors, the papers of Professor Wilder, Dr. Walsh, Miss Mullikin, Dr. Schweitzer, Dr. Rowe, Mr. Vandiver, Professor Lasley, Professor Moore, and Professor Birkhoff were read by title, and Professor Schwatt's papers were read by Professor H. H. Mitchell. Dr. Zeldin's second paper, Professor Kasner's first, Dr. Wiener's second, and Professor Forsyth's first paper were also read by title.

R. G. D. RICHARDSON,
Secretary.

* An interesting attempt at a direct demonstration of the theorem has been given recently by Mr. E. Gau, BULLETIN DES SCIENCES MATHÉMATIQUES vol. 43 (1919), p. 12–17. Unfortunately it is tacitly assumed (p. 14) that radial lines are carried over into curves met by any radial line only once.

RECIPROCAL SUBGROUPS OF AN ABELIAN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society September 8, 1920.)

§ 1. *Introduction.* Every two subgroups of the group G which have the property that the product of their orders is equal to the order of G have been called *reciprocal subgroups* of G.* A group may have subgroups which have no reciprocals. For instance, the tetrahedral group contains subgroups of order 2 but it does not contain any subgroup of order 6. If a group is abelian, each of its subgroups is known to have at least one reciprocal subgroup. A necessary and sufficient condition that every subgroup of G have one and only one reciprocal is that G be cyclic.

Two invariant subgroups of G will be called *corresponding reciprocal subgroups* of G if each of them is simply isomorphic with the quotient group of G with respect to the other. One of the objects of the present paper is to prove that every subgroup of an abelian group has a corresponding reciprocal subgroup. All the subgroups in a complete set of conjugate subgroups under the group of isomorphisms of G, that is, all the subgroups in a set of I-conjugate subgroups of G, must evidently have the same reciprocal subgroups. Hence the theory of corresponding reciprocal subgroups of an abelian group establishes a correspondence between pairs of sets of I-conjugate subgroups. In what follows it will be assumed that G is abelian.

As G is the direct product of its Sylow subgroups when the order of G is not a power of a prime number p and as the number of I-conjugates of a subgroup of such a G is the product of the numbers of the I-conjugates of the Sylow subgroups of this subgroup it will be assumed in what follows that the order of G is of the form p^m and that G has λ_1 invariants which are separately equal to p^{m_1}, λ_2 invariants which are separately equal to p^{m_2}, $\cdots$, λ_γ invariants which are separately equal to p^{m_γ}. Hence

$$\lambda_1 m_1 + \lambda_2 m_2 + \cdots + \lambda_\gamma m_\gamma = m.$$

It will be convenient to assume that $m_1 > m_2 > \cdots > m_\gamma$.

* This BULLETIN, vol. 9 (1903), p. 541.

Since every subgroup of index p^r contained in G contains the p^rth power of every operator of G, these powers constitute the cross-cut of all the subgroups of index p^r. The quotient group Q_r of G with respect to this cross-cut is simply isomorphic with the characteristic subgroup of G composed of all the operators of G whose orders divide p^r. This characteristic subgroup and the given cross-cut are characteristic corresponding subgroups of G. That is, the subgroup composed of the p^rth power of every operator of G and the subgroup composed of all the operators of G whose orders divide p^r are two characteristic corresponding subgroups of G.

Hence there is a (1, 1) correspondence between the subgroups of G which give rise to quotient groups involving no operators whose orders exceed p^r and the subgroups Q_r. In particular, there is a (1, 1) correspondence between the subgroups of G which give rise to quotient groups of type (1, 1, 1, $\cdots$) and the subgroups of the characteristic subgroup of G generated by all its operators whose orders divide p. Since the latter is known to contain as many subgroups of order p^r as it contains subgroups of index p^r we may state the following theorem.

THEOREM 1. *In every abelian group of order p^m the number of the subgroups of order p^r and of type* (1, 1, 1, $\cdots$) *is equal to the number of the subgroups of index p^r which separately give rise to a quotient group of type* (1, 1, 1, $\cdots$).

In this theorem r has an arbitrary value from 1 to the number of the independent generators of the abelian group in question.

The subgroups of index p^r which separately give rise to a quotient group of type (1, 1, 1, $\cdots$) may be of various types. Hence the theorem stated near the end of the preceding paragraph relates to an enumeration in which no distinction is made between subgroups of somewhat different types. It should be noted that the present method is based on relative properties of subgroups whose largest operators are of a given order and subgroups which give rise to quotient groups involving operators of this order but of no larger order.

If two reciprocal subgroups of G are such that their cross-cut is the identity then the sum of an arbitrary set of independent generators of one of these subgroups and an arbitrary set of independent generators of the other is a set of independent generators of G. Any two such reciprocal sub-

groups are corresponding reciprocal subgroups. A necessary and sufficient condition that two reciprocal subgroups of an abelian group generate this group is that their cross-cut is the identity.

§ 2. *Form of the Number of Subgroups of Certain Types.* It is well known that the number of the subgroups of a given order contained in G is always of the form $1 + kp$. As the subgroups of the same order may be of various types it is of interest to inquire whether there is any general theorem relating to the number of the subgroups of the same type. It is easy to prove that whenever G contains subgroups of the same order but of different types then the number of the subgroups of each of these types except one is divisible by p. That is, every abelian group of order p^m contains one and only one type of subgroups of order p^α, $\alpha < m$, such that the number of the subgroups of this type is of the form $1 + kp$.

To prove this theorem it is only necessary to observe that the number of the subgroups of a given type can be obtained by dividing the number of ways in which a set of independent generators of such a subgroup can be selected from the operators of G by the number of ways in which such a set can be selected from the operators of this subgroup. The numerator and the denominator of this quotient are commonly represented as the product of binomial factors. The first term of such a factor represents the order of the group generated by all the operators of the order in question contained in the group under consideration, while the second term represents the order of the subgroup of the former group composed of its operators which cannot be used as independent generators after the preceding independent generators, if any, have been chosen.

In order that the number of subgroups of a given type be of the form $1 + kp$ it must therefore be necessary that the second term of a factor of the given numerator is always the same as the corresponding second term in the denominator. This implies that the subgroup in question must have the property that if it contains operators of different orders it must involve all the operators of G of the same orders with the possible exception that the operators of highest order found in this subgroup need not include all the operators of the same order found in G. Whenever at least one of these second terms in

the numerator is larger than the corresponding second term in the denominator the number of subgroups is clearly divisible by p. Hence the following theorem has been established.

THEOREM 2. *Whenever an abelian group of order p^m contains subgroups of the same order but of different types then the number of the subgroups of one and of only one of these types is of the form $1 + kp$. The number of the subgroups of each of the other of these types is divisible by p.*

While G contains one and only one type of subgroups of each order which divides p^m such that the number of its subgroups of this type is of the form $1 + kp$ it may contain one or more than one type of subgroups of a given order such that the number of the subgroups of this type is a power of p. It is not difficult to determine a necessary and sufficient condition that the number of the subgroups of a given type be of the form p^e. In fact, one such condition is that each binomial factor of the numerator of the given quotient which represents the number of these subgroups is the product of a power of p by the corresponding factor in the denominator of this quotient. Hence we have the following theorem.

THEOREM 3. *A necessary and sufficient condition that the number of the subgroups of a given type contained in an abelian group G of order p^m be a power of p is that the number of its independent generators of each order increased by the number of its larger independent generators in this set be equal to the number of the independent generators of G whose orders are not less than this order.*

In the special case when there is only one subgroup of a given type the number of these subgroups may be said to be both a power of p and also of the form $1 + kp$. Hence in this special case the conditions involved in the two theorems stated above coincide. In order to illustrate the conditions under which the number of subgroups of a given type is a power of p we shall consider the special case when G is of type $(1, 2, 3, \cdots, m')$. To every combination of one or more of the numbers $1, 2, 3, \cdots, m'$ there corresponds one and only one such type, except the combination which involves all of these numbers. Hence this G contains $2^{m'} - 2$ different types of subgroups besides the identity such that the number of the subgroups of each of these types is a power of p. This is also the number of such types when G has equal invariants but when its distinct invariants are $p, p^2, p^3, \cdots, p^{m'}$.

In general, when the different invariants of G are p^{m_1}, p^{m_2}, $\cdots$, p^{m_λ}, it is not difficult to find the number of the different types of subgroups such that the number of the subgroups of each of these types is a power of p. In every one of the possible combinations of numbers $m_1, m_2, \cdots, m_\lambda$, each of these numbers may be replaced by every smaller integer which exceeds the one which follows it in this set. The sum of the sets thus obtained is the required number. In particular when G is of type $(m-1, 1)$ the number of such types of subgroups besides the identity is $2(m-2)$, and when all the invariants of G are equal to $p^{m'}$ this number is $m'-1$.

§ 3. *Quotient Groups and their Corresponding Subgroups.* It is well known that every possible quotient group of any abelian group is simply isomorphic with at least one subgroup of this abelian group, but two simply isomorphic subgroups of G do not always give rise to simply isomorphic quotient groups of G. Hence the question arises whether it is possible to associate with an arbitrary subgroup H_1 of G another subgroup H_2 of G such that G/H_1 is simply isomorphic with H_2 and G/H_2 is simply isomorphic with H_1, that is, whether for every subgroup of G there is at least one corresponding reciprocal subgroup.

For the sake of simplicity it will first be assumed that G/H_1 is cyclic, and it will be useful to note the following three possible cases: In the first case at least one operator in the co-set of G which corresponds to a generator of G/H_1 is of the same order as this generator. In the second case the ratios of the order of the smallest operators in a co-set and the order of the corresponding operator in G/H_1 are equal to the same number greater than unity for all the co-sets, excluding the co-set which corresponds to the identity in G/H_1. In the third case, the ratios of these orders are equal to $\rho > 1$ distinct numbers. The significance of conditions which are satisfied in these three cases can easily be determined and may be formulated as follows:

In the first case, any set of independent generators of H_1 together with an arbitrary operator of lowest order in a co-set corresponding to a generator of G/H_1 constitutes a set of independent generators of G. Conversely, whenever a set of independent generators of G can be so chosen that all of them except one generate H_1 then this case will present itself. In this case it is evident that H_1 and the cyclic subgroup gener-

ated by the remaining independent generator of G are corresponding reciprocal subgroups.

The second case implies that a set of independent generators of G can be so selected that all except one of them are independent generators of H_1 while the remaining independent generator of H_1 is a power of the remaining independent generator of G. This generator s is an arbitrary operator of lowest order in a co-set corresponding to a generator of G/H_1. Conversely, whenever a set of independent generators can be so selected that all except one of them are found in H_1 but do not generate H_1, then this second case will present itself. The cyclic subgroup of G which constitutes a corresponding reciprocal subgroup of H_1 is generated by the power of s which is simple isomorphic with G/H_1. It should be noted that the group generated by this power s' does not have only the identity in common with H_1; in fact it may be contained in H_1.

In the third case, the largest number of operators of any set of independent generators of G that can be selected from the operators of H_1 is ρ less than the total number of the independent generators of G, and the ratio of the orders of any two of these ρ generators cannot be less than p^2. On the other hand, whenever the ratio of the orders of any two of ρ generators of G is at least p^2 it is possible to find such a subgroup H_1. In fact, for the independent generators of H_1 we may take the independent generators of G exclusive of these ρ generators, plus the pth power of the largest of these ρ generators multiplied by the next in size, plus the pth power of this second multiplied by the next following in order of magnitude, $\cdots$, plus the pth power of the next to the last of these ρ operators multiplied by the last one. Hence we have the following theorem.

THEOREM 4. *If G is any abelian group of order p^m and ρ is the largest number of operators belonging to a set of independent generators of G and satisfying the condition that the ratio of the orders of any two of these ρ operators is not less than p^2, then G contains a subgroup H_1 which gives rise to a cyclic quotient group and satisfies the condition that at least ρ of the operators of every possible set of independent generators of G are not found in H_1.*

The main element of interest connected with this theorem is the fact that the lowest operators of G which correspond to the various operators of a cyclic quotient group G/H_1 together with the order of this quotient group determine completely

the set of I-conjugate subgroups to which H_1 belongs as well as the set of I-conjugate subgroups of G to which G/H_1 must correspond in order that H_1 and G/H_1 may be corresponding reciprocal subgroups. The ρ or $\rho - 1$ independent generators of H_1 which are not also independent generators of G can be obtained as follows.

First select the ρ independent generators of G $s_1, s_2, \cdots, s_\rho$ which are not contained in H_1 by noting that the first of these generators is any one of the operators of lowest order in the co-set of G which corresponds to a generator of G/H_1. The second is any operator of lowest order in the first lower co-set in which a power of s_1 is not an operator of lowest order. The third is any operator of lowest order in the next lower co-set in which a power of s_2 is not an operator of lowest order. . . . The last is any operator of lowest order in the highest co-set in which a power of $s_{\rho-1}$ is not an operator of lowest order. The independent generators of H_1 in question are then the product of s_2 into the inverse of the power of s_1 which occurs in the same co-set as s_2, the product of s_3 into the inverse of the power of s_2 which occurs in the same co-set as s_3, $\cdots$, the product of s_ρ by x the inverse of the power of $s_{\rho-1}$ which appears in the same co-set is s_ρ, the power of s_ρ which appears in H_1 whenever this power is not equal to the identity.

The cyclic subgroup of G which corresponds to G/H_1 may be obtained as follows: Let $s_1', s_2', \cdots, s_\rho'$ represent the powers of $s_1, s_2, \cdots, s_\rho$ respectively such that these powers are of the same orders as the operators of G/H_1 which correspond to $s_1, s_2, \cdots, s_\rho$ respectively. The product $s_1', s_2', \cdots, s_\rho'$ generates a cyclic subgroup H_1' which is simply isomorphic with G/H_1 and G/H_1' is also simply isomorphic with H_1.

When G/H_1 is non-cyclic it is the direct product of cyclic groups and the preceding arguments apply to these separate cyclic groups. At least one independent generator of G corresponds to each of these cyclic groups. Hence it results that in this case H_1 has again a corresponding reciprocal subgroup, and we have established the following theorem.

THEOREM 5. *Every subgroup of an abelian group has at least one corresponding reciprocal subgroup.*

THE UNIVERSITY OF ILLINOIS.

PROOF OF AN ARITHMETIC THEOREM DUE TO LIOUVILLE.

BY PROFESSOR E. T. BELL.

1. The theorem, quoted from volume II, page 337, of Dickson's *History of the Theory of Numbers*, is as follows. *If $f(m)$, $F(m)$ are two arbitrary functions having definite values for $m = 1, 2, 3, \cdots$, and*

$$X_\mu(m) = \Sigma d^\mu f(d), \qquad Z_\mu(m) = \Sigma d^\mu F(d),$$

where each summation extends over all divisors d of m, then for any real or complex numbers μ, ν, and $\delta = m/d$, we have

$$(A) \qquad \Sigma d^{\mu-\nu} X_\nu(d) Z_\mu(\delta) = \Sigma d^{\mu-\nu} Z_\nu(d) X_\mu(\delta).$$

No reference to a proof being given, presumably none has been published. But Liouville remarks* that "there is an exceedingly simple method which I shall develop on another occasion, and which will lead us very rapidly by a kind of regular and general algorithm to the formula (A)." Without going into more detail than suffices for the proof of (A) we shall indicate the nature of an algorithm of this type. Arbitrary functions such as f, F which take definite values for integral arguments are called numerical functions All of the functions considered in this paper are of this kind.

2. Let (d, δ) denote any pair of conjugate divisors of m. Form the value of $\varphi_1(x)\psi(y)$, where φ_1 and ψ are arbitrary functions, for $(x, y) = (d, \delta)$, sum $\varphi_1(x)\psi(y)$ over all pairs (d, δ), and denote the result by $\Sigma_m \varphi_1(d)\psi(\delta)$. Let (δ_1, δ_2) denote any pair of conjugate divisors of δ, so that $m = d\delta$, $\delta = \delta_1\delta_2$, $m = d\delta_1\delta_2$, and put $\psi(m) = \Sigma_m \varphi_2(d)\varphi_3(\delta)$; whence

$$\Sigma_m \varphi_1(d)\psi(\delta) = \Sigma_m [\varphi_1(d)\Sigma_\delta \varphi_2(\delta_1)\varphi_3(\delta_2)] = \Sigma_m \varphi_1(d_1)\varphi_2(d_2)\varphi_3(d_3)$$

the last summation extending over all triads (d_1, d_2, d_3) of divisors defined by $m = d_1 d_2 d_3$. If now (p, q, r), (i, j, k) are any permutations of $(1, 2, 3)$, it is obvious that

$$\Sigma_m [\varphi_p(d)\Sigma_\delta \varphi_q(\delta_1)\varphi_r(\delta_2)] = \Sigma_m \varphi_i(d_1)\varphi_j(d_2)\varphi_k(d_3),$$

which may be written in the purely symbolic form

$$\varphi_p \cdot \varphi_q \varphi_r = \varphi_i \varphi_j \varphi_k.$$

* Liouville, JOURNAL DES MATHÉMATIQUES, (2), vol. 3 (1858), p. 66.

This can be extended to any number of symbolic factors. Thus for four numerical functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, and (p, q, r, s), (i, j, k, l) any permutations of $(1, 2, 3, 4)$, we have

$$\varphi_p\varphi_q \cdot \varphi_r\varphi_s = \varphi_i \cdot \varphi_j\varphi_k\varphi_l = \varphi_i \cdot \varphi_j\varphi_k \cdot \varphi_l = \varphi_i\varphi_j\varphi_k\varphi_l, \text{ etc.}$$

In non-symbolic form, $\varphi_p\varphi_q \cdot \varphi_r\varphi_s$ is

$$\Sigma_m[\Sigma_d\varphi_1(d_1)\varphi_2(d_2)\Sigma_\delta\varphi_3(\delta_1)\varphi_4(\delta_2)],$$

the sums Σ_m, Σ_d, Σ_δ extending to all (d, δ), (d_1, d_2), (δ_1, δ_2), respectively, such that $m = d\delta$, $d = d_1d_2$, $\delta = \delta_1\delta_2$; and $\varphi_i\varphi_j\varphi_k\varphi_l$ is

$$\Sigma_m\varphi_i(d_1)\varphi_j(d_2)\varphi_k(d_3)\varphi_l(d_4),$$

the sum Σ_m extending to all tetrads (d_1, d_2, d_3, d_4) of divisors such that $m = d_1d_2d_3d_4$. The identity of these sums is obvious.

In precisely the same way the φ's in a product $\varphi_1\varphi_2 \cdots \varphi_n$ of any number n of symbolic factors are subject to the associative and commutative laws of formal algebra, and the interpretation of these laws for the general case in terms of multiple summations with respect to all the divisors of m distributed into sets of n or fewer each, is evident from the above illustrations.

3. Although it is not required for the proof of (A), we may point out that if each φ is such that $\varphi(1) \neq 0$, there is a unique division in this algebra. For it has been shown elsewhere* that for each φ there is a unique φ' satisfying the equations

$$\Sigma_1\varphi(d)\varphi'(\delta) = 1, \qquad \Sigma_m\varphi(d)\varphi'(\delta) = 0 \qquad (m > 1).$$

Expressing this symbolically we write $\varphi\varphi' = 1$, or $\varphi' = 1/\varphi$, and see that $\varphi_i\varphi_j = \varphi_k$ implies $\varphi_j = \varphi_k\varphi_i'$, or in another form, $\varphi_j = \varphi_k/\varphi_i$. When the arithmetical or algebraic definition of φ is given there is no difficulty in writing down the zeta function† which generates it. The zeta generator of φ' is obviously the reciprocal of the generator of φ.

* Bell, TÔHOKU MATHEMATICAL JOURNAL, vol. 17 (1920), p. 221.

† The zeta function generator of φ is by definition the Dirichlet series in which the coefficient of n^{-s} is $\varphi(n)$; *viz.* it is $\Sigma_{n=1}^{\infty}\varphi(n)/n^s$. For practically all the $\varphi(n)$ existing at present in arithmetic, s can be chosen so that the series converge absolutely, and for an important class of φ's, those in which $\varphi(mn) = \varphi(m)\varphi(n)$ for m, n relatively prime integers >0, the generator is decomposable into factors, etc., cf. Bachmann, *Zahlentheorie*, B. 2 Kap. 11. From these decompositions the relations between the φ's are developed isomorphically to the multiplicative properties of integers by means of the algorithm of this note. Such a course was evidently followed by Liouville in obtaining his results on numerical functions.

4. Write $u_k(m) = m^k$, and put

$$X_k'(m) = X_k(m)/m^k, \qquad Z_k'(m) = Z_k(m)/m^k.$$

Then from the definitions of the functions,

$$X_\mu' = u_{-\mu} f, \qquad Z_\mu' = u_{-\mu} F,$$

and from the associative and commutative laws,

$$u_{-\mu} F \cdot u_{-\nu} f = u_{-\mu} f \cdot u_{-\nu} F,$$

we find $Z_\mu' X_\nu' = X_\mu' Z_\nu'$, which may be written in full as follows:

$$\Sigma_m \frac{Z_\mu(\delta)}{\delta^\mu} \frac{X_\nu(d)}{d^\nu} = \Sigma_m \frac{X_\mu(\delta)}{\delta^\mu} \frac{Z_\nu(d)}{d^\nu}.$$

Multiplying this throughout by m^μ, we get (A).

THE UNIVERSITY OF WASHINGTON,
November 30, 1920.

A SEQUENCE OF POLYNOMIALS CONNECTED WITH THE nTH ROOTS OF UNITY.

BY DR. T. H. GRONWALL.

(Read before the American Mathematical Society September 7, 1920.)

In constructing examples of power series bounded in their circle of convergence and having specified convergence defects on the circle, it is frequently useful to consider polynomials of degree $n - 1$, such that at each of the nth roots of unity, the absolute value of the polynomial is less than or equal to a given constant M. Under these conditions, the maximum absolute value of the polynomial inside or on the unit circle is less than $4M \log n$.*

It is the purpose of this note to determine those polynomials where this maximum is as large as possible. The result may be stated in the following theorem.

THEOREM. *When the polynomial*

$$F(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$$

* E. Landau, *Bemerkungen zu einer Arbeit des Herrn Carleman*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 147–153.

has the property that

$$(1) \quad |F(\epsilon^\nu)| \leqq 1 \qquad (\nu = 0, 1, \cdots, n-1;\ \epsilon = e^{2\pi i/n})$$

and $n > 1$, *then*

$$(2) \quad |F(z)| < \frac{1}{n}\sum_{\nu=0}^{n-1} \frac{1}{\sin\dfrac{2\nu+1}{2n}\pi}, \qquad \text{for} \qquad |z| \leqq 1,$$

except when $F(z)$ *has the form* $e^{\alpha i}f(\epsilon^{-k}z)$, *where* α *is a real number,* k *is an integer, and*

$$(3) \quad f(z) = \sum_{\nu=0}^{n-1} \frac{(\epsilon^{-1/2}z)^\nu}{n\sin\dfrac{2\nu+1}{2n}\pi},$$

in which case the upper bound for $|F(z)|$ *is reached when* $z = \epsilon^{k+1/2}$. *The polynomial* $f(z)$ *has all its zeros on the unit circle, one in each of the intervals between two consecutive* n*th roots of unity, except the interval between* 1 *and* ϵ, *which contains no zero.*

The upper bound given on the right-hand side of (2) *is asymptotically equal to*

$$\frac{2}{\pi}\left(\log n + C + \log\frac{2}{\pi}\right) + o(1),$$

where C *is Euler's constant, and* $o(1)$ *tends to zero as* n *increases indefinitely.*

Let the absolute maximum of the absolute value of $|F(z)|$ for $|z| = 1$ occur between* ϵ^k and ϵ^{k+1}. If we write

$$F(z) = F_1(\epsilon^{-k}z),$$

the absolute maximum of $|F_1(z)|$ for $|z| = 1$ occurs between 1 and ϵ. By (1), we have for $\nu = 0, 1, \cdots, n-1$,

$$F_1(\epsilon^\nu) = M_\nu e^{\alpha_\nu i}, \qquad 0 \leqq M_\nu \leqq 1;$$

and since

$$g(z) = 1 + z + z^2 + \cdots + z^{n-1} = \frac{z^n - 1}{z - 1}$$

* If it occurs at an nth root of unity, then the maximum $|F(z)|$ is less than or equal to unity, which is less than the expression on the right in (2), each sine being less than unity, except the one corresponding to $\nu = (n-1)/2$ when n is odd.

is equal to n for $z = 1$, and is equal to zero for $z = \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$, we have, by Lagrange's interpolation formula,

$$nF_1(z) = \sum_{\mu=0}^{n-1} M_\mu e^{a_\mu i} g(\epsilon^{n-\mu} z). \tag{4}$$

Now

$$g(\epsilon^{n-\mu} e^{\theta i}) = \frac{e^{n\theta i} - 1}{e^{\left(\theta + \frac{n-\mu}{n} \cdot 2\pi\right) i} - 1}$$

$$= e^{\left(\frac{n-1}{2}\theta - \frac{n-\mu}{n}\pi\right) i} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2}\left(\theta + \frac{n-\mu}{n} \cdot 2\pi\right)},$$

and consequently, if we write $e^{\pi i/n} = \epsilon^{1/2}$,

$$nF_1(e^{\theta i}) = e^{\left(\frac{n-1}{2}\theta + a_0\right) i} \times \tag{5}$$

$$\times \left[M_0 \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} - \sum_{\mu=1}^{n-1} M_\mu e^{(a_\mu - a_0)i} \epsilon^{\mu/2} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2}\left(\theta + \frac{n-\mu}{n} \cdot 2\pi\right)} \right].$$

For $0 < \theta < 2\pi/n$, all the sines in this formula are obviously positive, so that for any value of θ in this interval, we have

$$n|F_1(e^{\theta i})| \leqq \frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} + \sum_{\mu=1}^{n-1} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2}\left(\theta + \frac{n-\mu}{n} \cdot 2\pi\right)},$$

where the equality sign holds when and only when we have $M_0 = M_1 = \cdots = M_{n-1} = 1$ and $e^{a_\mu i} = -\epsilon^{-\mu/2} e^{a_0 i}$, that is when $F_1(z) = e^{a_0 i} f_1(z)$, where, by (4) and (5),

$$nf_1(z) = g(z) - \sum_{\mu=1}^{n-1} \epsilon^{-\mu/2} g(\epsilon^{n-\mu} z), \tag{6}$$

$$nf_1(e^{\theta i}) = e^{\frac{n-1}{2}\theta i} \left[\frac{\sin \frac{n\theta}{2}}{\sin \frac{\theta}{2}} + \sum_{\mu=1}^{n-1} \frac{\sin \frac{n\theta}{2}}{\sin \frac{1}{2}\left(\theta + \frac{n-\mu}{n} \cdot 2\pi\right)} \right]. \tag{7}$$

Since $g(z) = 1 + z + \cdots + z^{n-1}$, the coefficient of z^ν in $nf_1(z)$ is, by (6),

$$1 - \sum_{\mu=1}^{n-1} \epsilon^{-\mu/2}\,\epsilon^{\nu(n-\mu)} = 1 - \frac{\epsilon^{-[(1/2)+\nu]} - \epsilon^{-[(1/2)+\nu]n}}{1 - \epsilon^{-[(1/2)+\nu]}}$$

$$= 1 - \frac{\epsilon^{-[(1/2)+\nu]} + 1}{1 - \epsilon^{-[(1/2)+\nu]}}$$

$$= -\frac{2}{\epsilon^{(1/2)+\nu} - 1} = \frac{i\epsilon^{-[(1/4)+(\nu/2)]}}{\sin\dfrac{2\nu+1}{2n}\pi}.$$

Consequently, defining $f(z)$ by (3) and α by $\alpha = \alpha_0 + \pi/2 - \pi/(2n)$, we find $f_1(z) = i\epsilon^{-1/4}f(z)$, and since the absolute maximum of $|f(z)|$ for $|z| = 1$ occurs when all terms to the right in (3) are positive, that is when $z = \epsilon^{1/2}$ or $\theta = \pi/n$ (which is the midpoint of the interval from 0 to $2\pi/n$), it follows that the absolute maximum for $|z| = 1$ of $|F(z)|$ is less than the right-hand member of (2), unless $F_1(z) = e^{\alpha_0 i}f_1(z) = e^{\alpha i}f(z)$, that is when $F(z) = e^{\alpha i}f(\epsilon^{-k}z)$.

The zeros of $f(z)$ are evidently those of $f_1(z)$, and (7) shows that $\varphi(\theta) = e^{-(n-1)\theta i/2}f_1(e^{\theta i})$ is real. Since, by (6), $f_1(1) = 1$, $f_1(\epsilon^\nu) = -\epsilon^{-\nu/2}$ for $\nu = 1, 2, \cdots, n-1$, it follows that $\varphi(0) = 1$, and $\varphi(2\nu\pi/n) = (-1)^{\nu-1}$ for $\nu = 1, 2, \cdots, n-1$, so that $\varphi(\theta)$ has an odd number of zeros in each of the intervals $2\nu\pi/n < \theta < (2\nu+2)\pi/n$, and consequently $f_1(z)$ has an odd number of zeros on the unit circle in each of the $n-1$ intervals from ϵ^ν to $\epsilon^{\nu+1}$ ($\nu = 1, 2, \cdots, n-1$). But $f_1(z)$ being of degree $n-1$ has exactly $n-1$ zeros, all of which therefore lie on the unit circle, one in each of the intervals mentioned. To find the asymptotic value of the expression to the right in (2), we observe that the νth and $(n-1-\nu)$th terms in the sum are equal, and that for n odd, there is a middle term equal to unity. Hence, for n even or odd,

$$\frac{1}{n}\sum_{\nu=0}^{n-1} \frac{1}{\sin\dfrac{2\nu+1}{2n}\pi} = \frac{2}{n}\sum_{\nu=0}^{[n/2]-1} \frac{1}{\sin\dfrac{2\nu+1}{2n}\pi} + o\left(\frac{1}{n}\right),$$

and by the definition of a definite integral

$$\lim_{n=\infty} \frac{\pi}{n} \sum_{\nu=0}^{[n/2]-1} \left(\frac{1}{\sin \frac{2\nu+1}{2n}\pi} - \frac{1}{\frac{2\nu+1}{2n}\pi} \right)$$

$$= \int_0^{\pi/2} \left(\frac{1}{\sin x} - \frac{1}{x} \right) dx = \log \frac{4}{\pi},$$

or

$$\frac{2}{n} \sum_{\nu=0}^{[n/2]-1} \frac{1}{\sin \frac{2\nu+1}{2n}\pi} = \frac{4}{\pi} \sum_{\nu=0}^{[n/2]-1} \frac{1}{2\nu+1} + \frac{2}{\pi} \log \frac{4}{\pi} + o(1).$$

Using the familiar asymptotic formula

$$\sum_{\nu=0}^{m} \frac{1}{2\nu+1} = \frac{1}{2} \log m + \frac{1}{2} C + o(1),$$

where C is Euler's constant, we find

$$\frac{1}{n} \sum_{\nu=0}^{n-1} \frac{1}{\sin \frac{2\nu+1}{2n}\pi} = \frac{2}{\pi} \left(\log n + C + \log \frac{2}{\pi} \right) + o(1).$$

TECHNICAL STAFF,
OFFICE OF THE CHIEF OF ORDNANCE

THE MINIMUM AREA BETWEEN A CURVE AND ITS CAUSTIC.

BY PROFESSOR PAUL R. RIDER.

(Read before the American Mathematical Society April 9, 1920.)

If rays from a given source of light are reflected by a curve, the envelope of the rays after reflection is called the caustic of the curve. It is an interesting problem to find the curve which connects two fixed points and which with its caustic and the rays reflected from the fixed points will enclose a minimum area. Euler* proposed and solved a similar problem

* Euler, *Methodus inueniendi lineas curvas maximi minimive proprietate gaudentes* or German translation in Ostwald's *Klassiker der exakten Wissenschaften*, no. 46. See also Todhunter, *Researches in the calculus of variations*, chapter 13.

concerning the curve which with its evolute and its normals at the fixed points will enclose the least area. The curves that furnish a solution of Euler's problem are cycloids. The curves that afford a solution of the problem concerning a curve and its caustic are transcendental curves of a more complex type. Their parametric equations are obtained in § 1. The determination of the arbitrary constants that occur in the solution is considered in § 2*. In § 3 a problem is solved which includes the evolute problem and the caustic problem as special cases.

§ 1. *Solution of the Problem.* We shall consider the case in which the source of light is at an infinite distance, and shall assume that the rays are parallel to the y-axis. Let the two fixed points be $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$, $x_0 \neq x_1$. We assume that the curve $y = y(x)$ which joins these two points and which with its caustic and the reflected rays at P_0 and P_1 encloses the minimum area, is single-valued with respect to x, and is continuous and possesses continuous derivatives.

If $\tau = \arctan y'$, the equation of the reflected ray from the point $(x, y(x))$ is readily found to be†

$$Y - y(x) = -\cot 2\tau \cdot (X - x),$$

or

$$Y - y(x) = \frac{y'^2(x) - 1}{2y'(x)}(X - x), \tag{1}$$

where X, Y are the current coordinates on the reflected ray.

Since the caustic is the envelope of the reflected rays, its parametric equations are obtained in the usual way by differentiating equation (1) with respect to the parameter x, and combining the new equation with (1). The equations of the caustic turn out to be

$$X = x - \frac{y'(x)}{y''(x)}, \quad Y = y(x) + \frac{1 - y'^2(x)}{2y''(x)}.$$

The distance from the point $(x, y(x))$ on the curve to the corresponding point X, Y on the caustic is

$$L = \frac{1 + y'^2(x)}{2y''(x)},$$

* Sufficient conditions, etc., will be treated in a later paper.

† Throughout the paper accents denote differentiation with respect to x.

and the differential of area between the curve and its caustic is

$$dA = \frac{1}{2} L \cos \tau \cdot ds = \frac{1 + y'^2(x)}{4y''(x)} dx,$$

where ds is the element of arc on the curve. Thus our problem reduces to that of minimizing the integral

$$A = \frac{1}{4} \int_{x_0}^{x_1} \frac{1 + y'^2(x)}{y''(x)} dx.$$

By the usual process of the calculus of variations* it is found that if the integral A is to be a minimum, $y(x)$ must satisfy the differential equation

$$\frac{2y'}{y''} + \frac{d}{dx} \frac{1 + y'^2}{y'^2} = \text{const.} = 2a. \tag{2}$$

Moreover, if the direction of the curve $y = y(x)$ at P_0 and P_1 is not prescribed, then we must have $(1 + y'^2)/y'' = 0$ at these points, that is the curve must have cusps at these points, and furthermore the caustic must pass through the points.

If we substitute $\tan \tau$ for y', we can easily reduce equation (2) to the form

$$-\frac{\cos^2 \tau \cdot \tau''}{\tau'^3} = a,$$

from which we find

$$-\frac{\tau''}{\tau'^2} = a \sec^2 \tau \cdot \tau'.$$

Integrating this equation, we get

$$\frac{1}{\tau'} = a \tan \tau + b,$$

or

$$dx = a \tan \tau d\tau + b d\tau.$$

This gives

$$x = a \log \sec \tau + b\tau + c,$$

or, if we set $p = y' = \tan \tau$,

$$x = a \log \sqrt{1 + p^2} + b \arctan p + c. \tag{3}$$

* See Bolza, *Vorlesungen über Variationsrechnung*, p. 152, Ex. 43.

Differentiating (3) with respect to y, we get

$$\frac{1}{p} = \frac{ap + b}{1 + p^2}\frac{dp}{dy},$$

from which we find that

$$y = b \log \sqrt{1 + p^2} - a \arctan p + ap + d.$$

Thus the parametric equations of the minimizing curves are

$$\begin{cases} x - c = a \log \sqrt{1 + p^2} + b \arctan p = a \log \sec \tau + b\tau, \\ y - d = b \log \sqrt{1 + p^2} - a \arctan p + ap \\ \qquad\qquad = b \log \sec \tau - a\tau + a \tan \tau. \end{cases}$$

The area between a curve of this set and its caustic is

$$A = \frac{1}{4}\int_{x_0}^{x_1} \frac{1 + y'^2}{y''}\,dx = \frac{1}{4}\int_{p_0}^{p_1} \frac{(ap + b)^2}{1 + p^2}\,dp$$

$$= \frac{a^2}{4}(p_1 - p_0) + \frac{ab}{2} \log \sqrt{\frac{1 + p_1^2}{1 + p_0^2}} + \frac{b^2 - a^2}{4}(\arctan p_1 - \arctan p_0)$$

$$= \frac{a^2}{4}(\tan \tau_1 - \tan \tau_0) + \frac{ab}{2} \log \frac{\sec \tau_1}{\sec \tau_0} + \frac{b^2 - a^2}{4}(\tau_1 - \tau_0).$$

§ 2. *Determination of the Constants.* If it is prescribed that the curve shall have the slope $p_0 = \tan \tau_0$ at the point $P_0(x_0, y_0)$ and the slope $p_1 = \tan \tau_1$ at the point $P_1(x_1, y_1)$, we have the following four equations from which to determine the constants a, b, c, d:

$$(4) \quad \begin{cases} x_i = a \log \sqrt{1 + p_i^2} + b \arctan p_i + c, \\ y_i = b \log \sqrt{1 + p_i^2} - a \arctan p_i + ap_i + d, \end{cases} \quad (i = 0, 1).$$

The determinant of this system is

$$\begin{vmatrix} \log \sqrt{1 + p_0^2} & \arctan p_0 & 1 & 0 \\ \log \sqrt{1 + p_1^2} & \arctan p_1 & 1 & 0 \\ p_0 - \arctan p_0 & \log \sqrt{1 + p_0^2} & 0 & 1 \\ p_1 - \arctan p_1 & \log \sqrt{1 + p_1^2} & 0 & 1 \end{vmatrix},$$

which reduces to

$$(p_1 - p_0)(\tau_1 - \tau_0) - (\tau_1 - \tau_0)^2 - (\log \sqrt{1 + p_1^2} - \log \sqrt{1 + p_0^2})^2.$$

Thus if

$$(p_1 - p_0)(\tau_1 - \tau_0) \neq (\tau_1 - \tau_0)^2 + (\log \sqrt{1 + p_1^2} - \log \sqrt{1 + p_0^2})^2,$$

the equations (4) can be solved for a, b, c, d.

§ 3. *A more general Problem.* If the integrand of the integral to be minimized is

$$f = (1 + p^2)^m q^n,$$

where $p = y'$, $q = y''$, we have a type of problem which includes the caustic problem for $m = 1$, $n = -1$, and Euler's evolute problem for $m = 2$, $n = -1$.

It is found* that the minimizing curve must be a solution of the differential equation

$$\frac{2m(1-n)p}{1+p^2} - \frac{n(1-n)q'}{q^2} = \frac{A}{(1+p^2)^m q^n}, \tag{5}$$

in which A is an arbitrary constant. If we multiply both sides of equation (5) by $dp = qdx$, we get

$$\frac{2m(1-n)p}{1+p^2}\,dp + \frac{n(1-n)}{q}\,dq = \frac{A}{(1+p^2)^m q^n}\,dp,$$

which reduces to

$$(1-n)d\log f = \frac{Adp}{f}.$$

Integration gives

$$(1-n)f = Ap + B.$$

Replacing f by $(1+p^2)^m q^n$ and solving for q (that is dp/dx), we obtain

$$\frac{dp}{dx} = \frac{1}{(1-n)^{1/n}} \frac{(Ap+B)^{1/n}}{(1+p^2)^{m/n}},$$

from which

$$x = (1-n)^{1/n} \int \frac{(1+p^2)^{m/n}}{(Ap+B)^{1/n}}\,dp. \tag{6}$$

* See Bolza, loc. cit.

Differentiating with respect to y, we get

$$\frac{1}{p} = (1-n)^{1/n} \frac{(1+p^2)^{m/n}}{(Ap+B)^{1/n}} \frac{dp}{dy},$$

and consequently

$$y = (1-n)^{1/n} \int \frac{(1+p^2)^{m/n} p}{(Ap+B)^{1/n}} dp. \tag{7}$$

Thus (6) and (7) are the parametric equations of the minimizing curves.

WASHINGTON UNIVERSITY,
ST. LOUIS, MO.,
October 12, 1920.

SHORTER NOTICES.

Il Problema dei Tre Corpi da Newton ai Nostri Giorni. By R. Marcolongo. (Manuali Hoepli.) Milan, Ulrico Hoepli, 1919. 162 pp.

This little book, in the well known style of the Hoepli manuals, presents, as its title indicates, an account of the problem of three bodies from the time of Newton to the present. The author has limited himself strictly to a descriptive account of what has been accomplished in this interval of time, with full references to original memoirs and papers where the interested reader can find the complete developments.

Professor Marcolongo is well known as an authority in the field of dynamical systems, and this book from his pen will be welcomed by all who are interested in the development of mathematical astronomy. Here will be found references to the works of over 200 authors who have contributed to one or more phases of this celebrated problem, together with a short description of the aim, the method of attack, and the results attained.

The book is divided into six chapters. The title of each sufficiently indicates its content.

I. The works of the geometers of the eighteenth century.
II. Reduction of the differential equations to least order.

III. Problem of n bodies. Particular cases.
IV. Uniform algebraic integrals in the problem of n bodies.
V. Approximate solutions by infinite series; by trigonometric series. Researches of Sundman.
VI. The restricted problem of three bodies. Periodic solutions. List of authors cited.

The American student will be interested to find among the authors mentioned the names of Hill, Newcomb, Longley, Brown, Lovett, Macmillan, Moulton, Birkhoff, Wilczynski; and to feel that his own country has not been behind in contributions to this special field of knowledge, important alike in its theoretical aspects and its practical bearings.

The complex development of modern mathematics calls for more books of this type: mathematical Baedeckers, without symbolism, with concise statements of aim, method of attack, and results, and with full references to original sources.

L. W. DOWLING.

Darstellende Geometrie. By Th. Schmid, associate professor of geometry at the technical school of Vienna. Volume I, second edition. (Sammlung Schubert, LXV.) Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1919. 278 pages and 170 figures.

THE first edition of Volume I appeared in 1912, and was reviewed in this BULLETIN (vol. 21 (1914), pp. 204–205.) We are told in the preface to this second edition that the second volume of the first edition has not yet been published. The manuscript is completed, but as the entire edition of the first volume was exhausted, it was decided to publish this second edition of the first volume before proceeding with the second volume.

In the present volume, the exercises appear in smaller type than the text, and are more numerous than in the former edition. At the end of § 12, marked § 35 in the new edition, paragraphs 1 and 2 are interchanged, and four pages of historical and bibliographical matter are added. A similar addition of two pages appears at the end of the volume. Otherwise it is almost a verbatim copy of the first edition.

VIRGIL SNYDER.

NOTES.

The date of the April meeting of the American Mathematical Society at New York has been changed by vote of the Council from April 30 to April 23. At this meeting Professor W. A. Hurwitz will present the opening paper of a symposium on divergent series.

The March number (vol. 22, no. 3) of the ANNALS OF MATHEMATICS contains the following papers: *The asymptotic expansion of the Sturm-Liouville functions*, by F. H. Murray; *On the conformal mapping of a region into a part of itself*, by J. F. Ritt; *Conjugate nets R and their transformations*, by L. P. Eisenhart; *The application of modern theories of integration to the solution of differential equations*, by T. C. Fry.

Volume 55 (1919–1920) of the PROCEEDINGS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES contains the following mathematical papers: *The functional relation of one variable to each of a number of correlated variables determined by a method of successive approximation to group averages. A contribution to statistical methods*, by G. F. McEwen and E. L. Michael; *Contribution to the general kinetics of material transformations*, by A. J. Lotka; *Rotations in space of even dimensions*, by H. B. Phillips and C. L. E. Moore; *Orbits resulting from assumed laws of motion*, by Arthur Searle; *Some geometric investigations on the general problem of dynamics*, by Joseph Lipka.

The October number (vol. 6, no. 10) of the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES contains: *On a condition for Helmholtz's equation similar to Lamé's*, by A. G. Webster; *Motion on a surface for any positional field of force*, by Joseph Lipka; The November number contains: *Seminvariants of a general system of linear homogeneous differential equations*, by E. B. Stouffer; *The permanent gravitational field in the Einstein theory*, by L. P. Eisenhart.

The following doctorates in mathematics were recently conferred by the University of London: H. E. J. Curzon: *The reversal of Halphen's transformation;* S. R. U. Saveer: *On the instability of the pear-shaped figure of equilibrium of a rotating mass of homogeneous liquid.*

The following persons have been appointed associate editors of the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY: Professors O. E. Glenn, A. J. Kempner, H. H. Mitchell, and J. H. M. Wedderburn.

The following persons have been appointed associate editors of the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY for the year 1921: Professors Dunham Jackson, Edward Kasner, D. N. Lehmer, Tullio Levi-Civita, and H. L. Rietz.

At the meeting of the Edinburgh Mathematical Society on December 10, 1920 the following papers were read: by G. D. Stokes, *Analytical treatment of the cam problem;* by D. G. Taylor, *Multiply perspective polygons inscribed in a plane cubic curve;* by T. A. Brown, *Converse of the Le Roy-Lindelöf theorem.* At the meeting on January 14, 1921, the following papers were read: By P. Humbert, *Some extensions of Pincherle's polynomials;* by T. M. MacRobert, *Some asymptotic expressions for the Bessel functions and the Fourier-Bessel expansions.*

At the meeting of the London Mathematical Society on December 9, 1920, the following papers were read: by S. Beatty, *The algebraic theory of algebraic functions of one variable;* by F. Debono, *The construction of magic squares;* by A. S. Eddington, *An application of the calculus of tensors to the theory of finite differences;* by A. R. Forsyth, *Developable surfaces through a couple of guiding curves in different planes;* by J. E. Jones, *The distribution of energy in the neighborhood of a vibrating sphere;* by L. J. Mordell, *I: The reciprocity formula for the Gauss's sums in a quadratic field; II: A new class of definite integrals;* by G. N. Watson, *The product of two hypergeometric functions;* by W. H. Young, *I: Integration over the area of a surface and transformation of the variables in a multiple integral; II: A new set of conditions for a formula for an area.* At the meeting on January 13, 1921, papers were read by A. S. Eddington, *Dr. Sheppard's method of reduction of error by linear compounding;* by W. F. Sheppard, *Conjugate sets of quantities;* by E. A. Milne, *A problem concerning the maxima of certain types of sums and integrals;* by H. J. Priestley, *The linear differential equation of the second order;* by M. Kössler, *The zeros of analytic functions;* by A. C. Dixon, *The theory of a thin elastic plate, bounded by two circular arcs, and clamped;* by G. A. Miller, *Determination of all the characteristic sub-groups of an abelian group.*

The following courses in mathematics are announced:

COLUMBIA UNIVERSITY (summer session, July 5 to August 12).—By Professor Edward Kasner: General survey of modern mathematics, five hours; Mathematical introduction to Einstein's theory of relativity, five hours.—By Professor W. B. Fite: Differential equations, five hours.—By Dr. G. A. Pfeiffer: Theory of functions of a real variable, five hours.—By Dr. J. F. Ritt: Theory of numbers, five hours.

COLUMBIA UNIVERSITY (academic year 1921–1922).—By Professor T. S. Fiske: Differential equations, four hours.—By Professor F. N. Cole: Invariants and higher plane curves, three hours (second term); Theory of groups, three hours.—By Professor D. E. Smith: History of mathematics, two hours; Practicum in the history of mathematics, four hours.—By Professor C. J. Keyser: Modern theories in geometry, four hours; Introduction to mathematical philosophy, two hours (first term).—By Professor Edward Kasner: Einstein's theory of gravitation, two hours.—By Professor W. B. Fite: Infinite series, three hours (first term); Calculus of variations, three hours (second term).—By Dr. G. A. Pfeiffer: The theory of sets of points, three hours.—By Dr. J. F. Ritt: Functional equations, three hours.

CORNELL UNIVERSITY (summer session, July 9 to August 19).—By Professor V. Snyder: Projective geometry, six hours.—By Professor W. A. Hurwitz: Analysis, six hours.

CORNELL UNIVERSITY (academic year 1921–1922).—By Professor J. H. Tanner: Mathematics of finance, two hours (given in each term).—By Professor F. R. Sharpe: Hydronamics and elasticity, three hours.—By Professor W. B, Carver: Modern algebra, three hours.—By Professor A. Ranum: Non-euclidean geometry, three hours.—By Professor D. C. Gillespie: Advanced calculus, three hours.—By Professor W. A. Hurwitz: Linear integral and differential equations, three hours.—By Professor C. F. Craig: Functions of a complex variable, three hours.—By Professor F. W. Owens: Theory of probability, three hours.—By Dr. H. M. Morse: Elliptic functions, three hours.—By Dr. Helen B. Owens: Projective geometry, three hours.—By Dr. F. W. Reed: Dynamics, three hours.—By Dr. G. M. Robison: Differential equations, three hours.—By Mr. H. S. Vandiver: Theory of groups, three hours.—By Dr. W. L. G. Williams: Advanced analytic geometry, three hours.

PRINCETON UNIVERSITY (academic year 1921–1922).—By Professor H. B. Fine: Functions of a complex variable, three hours.—By Professor O. Veblen: Projective geometry, three hours.—By Professor J. W. Alexander: Differential equations, three hours. Professors Eisenhart and Veblen will conduct a seminar in relativity.

Professor R. Courant, of the University of Münster, has been called to the chair of mathematics at the University of Göttingen that was vacated by Professor Felix Klein on his retirement.

Professor P. R. Scott Lang, for more than forty years Regius professor of mathematics at the University of St. Andrews, has been knighted.

Professor W. W. Rankin, of the University of North Carolina, who has been on leave of absence for the last two years acting as instructor at Columbia University, has been appointed head of the department of mathematics at Agnes Scott College.

At Cornell University, Professor V. Snyder has been granted leave of absence for the academic year 1921–1922. For the second term of the year he has been awarded a subvention by the Heckscher Research Foundation for the prosecution of research on the theory of algebraic surfaces.

Professor Arthur Searle, Phillips professor of astronomy, emeritus, at Harvard University, died October 24, 1920, at the age of eighty-three years.

Dr. Alexander Pell, formerly professor of mathematics and astronomy and dean of the engineering school at the University of South Dakota, and associate professor at Armour Institute, died January 26, 1921, at the age of sixty-two years. Dr. Pell had been a member of the American Mathematical Society since 1898.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDERSEN (A. F.), BOHR (H.) og MOLLERUP (J.). Cesàro's Summabilitetsmetode med Anvendelse paa Fourier'ske og Dirichlet'ske Rækker. Tre Foredrag holdt i Matematisk Forening (København). København, Gjellerup, 1919. 8vo. 51 + 20 + 25 pp.

BOHR (H.). See ANDERSEN (A. F.).

BOUTROUX (P.). L'idéal scientifique des mathématiciens. Paris, Alcan, 1920. 16mo. 274 pp.

BRENDEL (M.). See KLEIN (F.).

BUHL (A.). Géométrie et analyse des intégrales doubles. (Scientia, No. 36.) Paris, Gauthier-Villars, 1920. 68 pp.

CURZON (H. E. J.). The reversal of Halphen's transformation. London, Constable, 1920. 15 pp. 1s.

FRAENKEL (A.). See KLEIN (F.).

FRASER (D. C.). Newton's interpolation formulas. London, C. and E. Layton (published under the authority of the Institute of Actuaries), 1919. 52 pp. 1s.

HEIJDEMAN (W. J.). Wiskundige hoofstukken dienende als noodzakelijke inleiding tot de differentiaal- en integraalrekening voor den technicus, die de hoogere wiskunde door zelfstudie wil leeren; tevens bevattende de eigenschappen en constructies der kromme lijnen en oppervlakken, welke voor den technicus van belang zijn. Deventer, Kluwer, 1919. 140 pp.

KLEIN (F.), BRENDEL (M.) und SCHLESINGER (L.). Materialien für eine wissenschaftliche Biographie von Gauss. Heft 8: Zahlbegriff und Algebra bei Gauss, von A. Fraenkel. Mit einem Anhang von A. Ostrowski: Zum ersten und vierten Gaussschen Beweise des Fundamentalsatzes der Algebra. Leipzig, Teubner, 1920. 8vo. 60 pp.

LORIA (G.). Newton. Roma, A. F. Formíggini, 1920. 69 pp. L. 3.00

MACH (E.). Die Leitgedanken meiner naturwissenschaftlichen Erkenntnislehre und ihre Aufnahme durch die Zeitgenossen. Sinnliche Elemente und naturwissenschaftliche Begriffe. Zwei Aufsätze. Leipzig, Barth, 1919. 8vo. 31 pp.

MOLLERUP (J.). See ANDERSEN (A. F.).

OSTROWSKI (A.). See KLEIN (F.).

PASCAL (E.). Lezioni di calcolo infinitesimale. Parte 3: Calcolo delle variazioni e delle differenze finite. 2a edizione. (Manuali Hoepli.) Milano, Hoepli, 1918. 11 + 325 pp.

RIABOUCHINSKY (D.). Calcul des valeurs absolues. Copenhagen, 1919. 119 pp.

SCHLESINGER (L.). See KLEIN (F.).

SCHLICK (M.). Allgemeine Erkenntnislehre. Berlin, Springer, 1918. 8vo. 346 pp.

SIBIRANI (F.). See VIVANTI (G.).

VIVANTI (G.). Esercizi di analisi infinitesimale. 2a edizione, accuratamente riveduta ed aumentata con la collaborazione del professore F. Sibirani. Parte 1a. Torino, Lattes, 1920. 7 + 172 pp.

VOLK (O.). Entwicklung der Funktionen einer komplexen Variabeln nach den Funktionen des elliptischen Zylinders. Stuttgart, Grub, 1920. 38 pp.

WATSON (G. N.). See WHITTAKER (E. T.).

WHITTAKER (E. T.) and WATSON (G. N.). A course of modern analysis. 3d edition. Cambridge, University Press, 1920. 7 + 608 pp. 40s.

WITTING (A.). Einführung in die Infinitesimalrechnung. I: Differentialrechnung. 2te Auflage. (Mathematisch-physikalische Bibliothek, Nr. 9.) Leipzig, Teubner, 1920.

II. ELEMENTARY MATHEMATICS.

ARNOLD (E. E.). See DURELL (F.).

BINGHAM (J. A.). See HOWARD (B. H.).

CRELLE (A. L.). Rechentafeln, welche alles Multiplizieren und Dividieren mit Zahlen unter Tausend ganz ersparen, bei grösseren Zahlen aber die Rechnung erleichtern und sicherer machen. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919.

DAVISON (C.). The elements of plane geometry. Cambridge, University Press, 1920. 280 pp. 10s.

DOOLEY (W. H.). Vocational mathematics. Revised by A. Ritchie-Scott. London, Heath, 1920. 8 + 311 pp. 5s.

DURELL (F.) and ARNOLD (E. E.). A second book in algebra. Chicago, Merrill, 1920. 5 + 330 pp.

HAMILTON (S.). Hamilton's essentials of arithmetic. Lower grades, middle grades, higher grades. Chicago, American Book Company, 1920. 224 + 14, 288 + 23, 320 + 19 pp.

HOWARD (B. H.) and BINGHAM (J. A.). A school geometry. London, Hodder and Stoughton, 1920. 26 + 370 pp. 5s. 6d.

LAYNG (A. E.). Exercises in arithmetic. Arranged in two courses. London, John Murray, 1920. 230 + 31 pp. 3s. 6d.

LEVENTHAL (M. J.) and WEINER (M.). Geometry review book. New York, Review Book Company, 1920. 86 pp.

LINCOLN SCHOOL. Illustrated mathematical talks by pupils of the Lincoln School, New York City. New York, Lincoln School, 1920. 44 pp.

PICKEN (D. K.). See WADDELL (W.).

RITCHIE-SCOTT (A.). See DOOLEY (W. H.).

WADDELL (W.) and PICKEN (D. K.). A first trigonometry. Melbourne, Melville and Mullin, 1920. 7 + 78 pp.

WEINER (M.). See LEVENTHAL (M. J.).

III. APPLIED MATHEMATICS.

ADAMS (O. S.). A study of map projections in general. (U. S. Coast and Geodetic Survey, Special Publication No. 60.) Washington, 1919. 24 pp.

APPELL (P.). Traité de mécanique rationnelle. Tome III: Equilibre et mouvement des milieux continus. 3e édition, entièrement refondue. Tome IV: Figures d'équilibre d'une masse fluide homogène en rotation sous l'attraction newtonienne de ses particles. Paris, Gauthier-Villars, 1920. 8vo. 7 + 646 + 297 pp. Fr. 60.00 + 30.00

ASHFORD (C. E.). Electricity and magnetism; theoretical and practical. 3d edition. London, Arnold, 1920. 12 + 303 pp. 4s. 6d.

BLACKBURN (P. P.). See WHITE (C. J.).

BOUASSE (H.). Pendule, spiral, diapason. Tome I. Paris, Delagrave, 1920. 8vo. 26 + 476 pp.

COTTON (E.). Cours de mécanique générale. Introduction à l'étude de la mécanique industrielle. Tome III: Unités. Travail. Dynamique du point et des systèmes. Grenoble, J. Rey, et Paris, Gauthier-Villars, 1920. 8vo. 138 pp.

DESMARETS (M.). See MALGORN (G.).

DINGLER (H.). Die Grundlagen der Physik. Synthetische Prinzipien der mathematischen Naturphilosophie. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1919. 8vo. 15 + 157 pp.

EILERS (G.). Am Schattenstab. Eine volkstümliche Himmelskunde in geschichtlicher Anordnung. Hamburg, Westermann, 1920. 192 pp.

EINSTEIN (A.). Aether und Relativitäts-Theorie. Rede gehalten am 5. Mai an der Reichs-Universität zu Leiden. Berlin, Springer, 1920. 15 pp.

ELMASSIAN (P.). L'éther pur, l'éther matériel et les trois formes fondamentales de l'énergie. Genève, Atar, 1920. 304 pp.

GANS (R.). Einführung in die Vektoranalysis mit Anwendungen auf die mathematische Physik. 4te Auflage. Leipzig, Teubner, 1921. 4 + 118 pp.

HAAS (A.). Das Naturbild der neuen Physik. Berlin, Vereinigung wissenschaftlicher Verleger (Walter de Gruyter), 1920. 5 + 114 pp.

KLEEMAN (R. D.). A kinetic theory of gases and spectra. New York, Wiley, 1920. 16 + 272 pp. $3.00

LEROY (T.). Essai mathématique sur les prix de revient des transports par chemin de fer. Préface de G. Pereire. Paris, Gauthier-Villars, 1920. 4to. 16 + 246 pp. Fr. 30.00

LEWIS (W. C.). A system of physical chemistry. Volume 2: Thermodynamics. 3d edition. London, Longmans, 1920. 8 + 454 pp. 15s.

MALGORN (G.). Lexique technique anglais-français. Machines-outils, moteurs à combustion interne, électricité, constructions navales, métallurgie, etc. Avec la collaboration de M. Desmarets. Paris, Gauthier-Villars, 1920. 22 + 216 pp. Fr. 20.00

PARK (W. E.). A treatise on airscrews. London, Chapman and Hall, 1920. 12 + 308 pp. 21s.

PEREIRE (G.). See LEROY (T.).

POMEY (J. B.). Introduction à la théorie des courants téléphoniques et de la radiotélégraphie. Paris, Gauthier-Villars, 1920. 8vo. 510 pp. Fr. 25.00

STELFOX (S. H.). The laws of mechanics. A supplementary text-book. London, Methuen, 1920. 9 + 201 pp. 6s.

WHITE (C. J.). The elements of theoretical and descriptive astronomy. 8th edition, revised by P. P. Blackburn. New York, Wiley, 1920. 11 + 309 pp. $3.00

WILLIAMS (K. P.). The dynamics of the airplane. (Mathematical Monographs, No. 21.) New York, Wiley, 1921. 8 + 138 pp. $2.50

WILLOTTE (H.). Lois mathématiques de la résistance des fluides. (Théorie de l'hélice.) Paris, Doin, 1920. 16mo. 300 pp.